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Phase-Plane Solutions of Langmuir's Equation*

ALFRED INSELBERG

IBM Scientific Center, 1930 Century Park West, Los Angeles, California 90067

Submitted by Richard Bellman

1. INTRODUCTION

In 1923 Langmuir's investigations [1] into the theory of the flow of current from a hot cathode to a positively charged anode in high vacuum led to the equation

$$3y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 4y \frac{dy}{dx} + y^2 = 1. \tag{1}$$

A more convenient form [2],

$$3z \frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2 - e^x = 0, \tag{2}$$

is obtained by means of the transformation $y = ze^{-x/2}$.

Though Langmuir's equation is of considerable interest, little is known about its properties since both forms, Eq. (1) and Eq. (2) are difficult to study.

In the next section two new forms of Eq. (1) are derived from which, in Section 3, asymptotic phase-plane solutions in closed form are obtained.

A study of superposition principles for nonlinear operators motivated this work [3]. In particular, a canonical form for a number of ordinary non-linear differential operators, T, which appear in the applications is

$$T(u) = h(u)\, g^{-1}[L(f(u))], \tag{3}$$

where h, g and f are functions in the domain of T, and L is a linear differential operator. Examples of operators having such a representation are also provided in [4]. The form given in (3) may be defined in a more general context which, however, is not necessary for the purpose of this paper.

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2. The new forms

Note that in Eq. (1) if the first two terms had the same coefficient, say a, the equation could be put into the form

$$L(y^2) = 1 \tag{4}$$

where

$$L(y) = \frac{a}{2}\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y,$$

an equation whose solution is readily obtained. Lamentably, this is not the case for Langmuir's equation, though a representation similar to (4) may be obtained. In particular, h, f and L are sought such that

$$\Lambda(y) = 3y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 4y\frac{dy}{dx} + y^2 = h(y)\,L(f(y)) \tag{5}$$

where

$$L(y) = a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)\,y.$$

Notice that the g in (3) is here assumed to be the identity transformation.

Proceeding by expanding the right-hand side of (5) leads to

$$h(y)\,L(f(y)) = h(y)\left[a_2\left\{\frac{d^2f}{dy^2}\left(\frac{dy}{dx}\right)^2 + \frac{df}{dy}\frac{d^2y}{dx^2}\right\} + a_1\frac{df}{dy}\frac{dy}{dx} + a_0 f\right]. \tag{6}$$

Matching the coefficients of like terms results in the following relations:

$$a_2(x)\,h(y)\frac{d^2f}{dy^2} = 1 \tag{7}$$

$$a_2(x)\,h(y)\frac{df}{dy} = 3y \tag{8}$$

$$a_1(x)\,h(y)\frac{df}{dy} = 4y \tag{9}$$

$$a_0(x)\,h(y)\,f = y^2. \tag{10}$$

It is immediately seen that the $a_i(x)$ $(i = 0, 1, 2)$ must be constant since f and h are functions of y only. Let

$$a_i(x) = k_i\,, \qquad i = 0, 1, 2, \tag{11}$$

with

$$k_2 = \tfrac{3}{4}\,k_1 \tag{12}$$

obtained from (8) and (9).

Dividing Eq. (7) by Eq. (8) provides the equation

$$\frac{df'}{f'} = \frac{dy}{3y} \qquad \left[f' = \frac{df}{dy}\right]$$

which when integrated yields

$$f'(y) = cy^{1/3}.$$

Hence

$$f(y) = y^{4/3} \tag{13}$$

with the arbitrary constant c set, for convenience at $\frac{4}{3}$.

The function h is obtained from (10) and (13) and is

$$h(y) = \frac{y^{2/3}}{k_0}. \tag{14}$$

There remains the evaluation of the constants k_i.

Substituting (13) and (14) into (9) shows that

$$k_1 = 3k_0 \tag{15}$$

which together with (12) constitute the set of independent constraints for the k's. Assigning k_0 the value of one leads to

$$k_0 = 1, \qquad k_1 = 3 \qquad \text{and} \qquad k_2 = \tfrac{9}{4}. \tag{16}$$

These calculations then show that Langmuir's equation may be written as

$$\Lambda(y) = y^{2/3}L(y^{4/3}) = 1, \tag{17}$$

with

$$L(y) = \frac{9}{4}\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y.$$

By a fortunate coincidence L turns out to be a perfect square namely,

$$L(y) = (\tfrac{3}{2}D + 1)^2 y,$$

Where as usual $D = d/dx$.

Therefore,

$$\Lambda(y) = \tfrac{9}{4}y^{2/3}(D + \tfrac{2}{3})^2 y^{4/3} = 1. \tag{18}$$

The symmetry between the exponents of y and the exponent and the translation constant of the differential operator in Eq. (18) is rather remarkable. This unusual algebraic structure may well provide clues about the properties of Langmuir's equation. Perhaps an inspired reader may find ways of utilizing the symmetry to this end.

From Eq. (18) the two new forms may be immediately obtained.

Recalling the translation identity

$$(D + r)^k u = e^{-rx} D^k(e^{rx}u)$$

and applying to Eq. (18), by means of the transformation

$$u = e^{1/3x} y^{2/3}, \tag{19}$$

yields

$$u \frac{d^2u^2}{dx^2} = \frac{4}{9} e^x. \tag{20}$$

This equation has a form distinctly superior to Eq. (1) since it involves only one nonlinear term. Notice that $u = e^{1/3x}$ is a solution of Eq. (20) which corresponds to the singular solutions $y = \pm 1$ of Eq. (1).

A different approach may also be pursued with Eq. (18). Since this, as well as Eq. (1), is an autonomous equation, employ

$$p = \frac{dv}{dx}, \qquad \text{with} \qquad v^{1/2} = y^{2/3}, \tag{21}$$

to reduce the order by one obtaining

$$p \frac{dp}{dv} + \frac{4}{3} p = \frac{4}{9} (v^{-1/2} - v). \tag{22}$$

Eq. (22) is seen to be an Abel equation of the second kind [5], and can be put into a more compact form by letting

$$p = q - \tfrac{4}{3} v \tag{23}$$

to obtain

$$\left(q - \frac{4}{3} v\right) \frac{dq}{dv} = \frac{4}{9} (v^{-1/2} - v).$$

Finally let

$$z = v^{1/2} \text{ (positive branch)} \tag{24}$$

to get

$$\left(q - \frac{4}{3} z^2\right) \frac{dq}{dz} = \frac{8}{9} (1 - z^3). \tag{25}$$

The connection between Langmuir's equation and Abel's equation, Eq. (25), is exploited in the next section.

3. Phase-Plane Solutions

The elegant formulations of Langmuir's equation are Eqs. (18) and (20), though unfortunately, the attempts to study these equations have not been fruitful thus far. On the other hand, Eq. (25) yields asymptotic solutions by some fairly direct methods as shown in this section.

In Eq. (25) the forcing term is such that for small values of z (say $|z| < \frac{1}{2}$) the constant term dominates while for large z (say $|z| > 2$) the cubic term dominates. For values of z close to one the forcing term becomes very small. Guided by these observations the solutions are now obtained in these three different regions.

For large $|z|$ Eq. (25) is approximated by

$$\left(q - \frac{4}{3} z^2\right) \frac{dq}{dz} = -\frac{8}{9} z^3. \tag{26}$$

Letting

$$w(q) = \tfrac{4}{3} z^2 \tag{27}$$

in Eq. (26) results in

$$ww' - 4w + 4q = 0, \tag{28}$$

an equation whose exact solution is known [6],

$$(w - 2q) \exp \frac{-2q}{(w - 2q)} = C \tag{29}$$

or, in terms of v and v',

$$\left(v' + \frac{2}{3} v\right) \exp \frac{(y' + \frac{4}{3} v)}{(v' + \frac{2}{3} v)} = -\frac{C}{2} = k. \tag{30}$$

A more convenient parametric representation for the phase-plane solution is provided by letting

$$\alpha = v' + \tfrac{2}{3} v \tag{31}$$

in Eq. (30), yielding

$$v = \left[\ln \left(\frac{k}{e\alpha}\right)^{3/2\alpha}\right], \qquad v' = \ln \left(\frac{\alpha e^2}{k}\right)^{\alpha}. \tag{32}$$

The extrema of y are easily obtained from Eq. (30) by setting $y' = 0$. They are at

$$y = \left(\frac{3}{2} \frac{k}{e^2}\right)^{3/4}. \tag{33}$$

For small $|z|$ Eq. (25) is approximated by

$$\left(q - \frac{4}{3} z^2\right) \frac{dq}{dz} = \frac{8}{9}. \tag{34}$$

Letting

$$w(q) = z$$

in Eq. (34) results in

$$w' + \tfrac{3}{2} w^2 - \tfrac{9}{8} q = 0. \tag{36}$$

This is a Riccati equation which may be linearized into

$$t'' - \frac{27}{16} qt = 0, \tag{37}$$

by means of the transformation

$$w(q) = \frac{2}{3} \frac{t'(q)}{t(q)}. \tag{38}$$

Eq. (37) may be converted into a Bessel equation, namely

$$r^2 \psi''(r) + r\psi'(r) - (\tfrac{1}{9} + r^2)\,\psi(r) = 0 \tag{39}$$

with

$$t(q) = \varphi(s), \qquad s = \frac{3}{2^{4/3}} q \tag{40}$$

$$\varphi(s) = \psi(r)\sqrt{s}\,, \qquad r = \tfrac{2}{3} s^{3/2} \tag{41}$$

being the intermediate steps.

The solution of Eq. (39) is

$$\psi(r) = c_1 I_{1/3}(r) + c_2 I_{-1/3}(r) \tag{42}$$

where I_p is the modified Bessel function of the first kind and of order p.

Reversing the steps yields

$$t(q) = \frac{\sqrt{3}}{2^{2/3}} q^{1/2} \left\{ c_1 I_{1/3}\left(\frac{\sqrt{3}}{2} q^{3/2}\right) + c_2 I_{-1/3}\left(\frac{\sqrt{3}}{2} q^{3/2}\right) \right\}, \tag{43}$$

an expression resembling Airy's function.

Using the formula [7]

$$I_p'(z) = \tfrac{1}{2}\left(I_{p-1}(z) + I_{p+1}(z)\right)$$

$w(q)$ is obtained as

$$w(q) = \frac{1}{3q} + \frac{\sqrt{3}}{4} q^{1/2} \frac{\left[\begin{aligned} &\left\{ I_{-2/3}\left(\frac{\sqrt{3}}{2} q^{3/2}\right) + I_{4/3}\left(\frac{\sqrt{3}}{2} q^{3/2}\right) \right\} \\ &\quad + C \left\{ I_{-4/3}\left(\frac{\sqrt{3}}{2} q^{3/2}\right) + I_{2/3}\left(\frac{\sqrt{3}}{2} q^{3/2}\right) \right\} \end{aligned} \right]}{I_{1/3}\left(\frac{\sqrt{3}}{2} q^{3/2}\right) + C I_{-1/3}\left(\frac{\sqrt{3}}{2} q^{3/2}\right)} \tag{44}$$

with $w(q) = v^{1/2}$

$$q - \frac{4}{3} v = v' \qquad \text{and} \qquad C = \frac{c_2}{c_1}. \tag{45}$$

It is fascinating to discover that Langmuir's equation is connected with the earlier works of Abel, Riccati, Bessel and Airy.

A great deal of difficulty arises in the investigation of the solution of Eq. (25) in the neighborhood of $z = 1$. It is instructive to see why. Let

$$z = 1 + \epsilon \tag{46}$$

with ϵ small and substitute in Eq. (25) neglecting ϵ^3 to obtain

$$\left[q - \frac{4}{3}(1+\epsilon)^2\right]\frac{dq}{dz} = -\frac{8}{3}\epsilon(1+\epsilon).$$

Adding and subtracting $\frac{8}{3}(1+\epsilon)$ to the right-hand member results in

$$\left(q - \frac{4}{3}z^2\right)\frac{dq}{dz} = \frac{8}{3}(z - z^2). \tag{47}$$

Employing the transformation

$$m(q) = z$$

leads to

$$2(m - m^2)\, m' + m^2 = \tfrac{3}{4} q$$

which may also be written as

$$e^{2m}(D_q + 1)\,(e^{-2m}m^2) = \tfrac{3}{4} q,$$

or equivalently

$$e^{2m-q}\, D_q(e^{-2m+q}m^2) = \tfrac{3}{4} q.$$

It has not been possible thus far to solve Eq. (47) or its alternate forms.

An equation more tractable than Eq. (47) results when a rougher but still fairly accurate approximation is introduced in Eq. (25). In particular for z as in Eq. (46)

$$\left(u - \frac{8}{3}\epsilon\right)\frac{du}{d\epsilon} = -\frac{8}{3}\epsilon(1+\epsilon) \tag{48}$$

with

$$u = q - \tfrac{4}{3}.$$

The transformation

$$u - \frac{8}{3}\epsilon = \frac{1}{w}$$

converts Eq. (48) to

$$w' = \tfrac{8}{3} w^2 + \tfrac{8}{3} \epsilon(1 + \epsilon) w^3, \tag{49}$$

an Abel equation of the first kind. This equation is related to the Lane-Emden equation [8]. The connection is, however, rather involved and only partial information regarding the solution of Eq. (48) is obtained in this way.

These difficulties result from the fact that at $|z| = 1$ the singular solutions, $y = \pm 1$, of the original Langmuir equation occur. However, ignoring the nonlinear terms in z^2 and z^3 for $|\epsilon| \leqslant \frac{1}{4}$, still provides a useful approximation and also results in an equation that can be solved, namely:

$$\left(q - \frac{4}{3} - \frac{8}{3}\epsilon\right)\frac{dq}{d\epsilon} = -\frac{8}{3}\epsilon. \tag{50}$$

By letting

$$u = q - \tfrac{4}{3} - \tfrac{8}{3}\epsilon \tag{51}$$

Eq. (50) becomes

$$uu' + \tfrac{8}{3} u + \tfrac{8}{3}\epsilon = 0, \tag{52}$$

whose solution [9] is given by

$$\ln X_1 = C + \tfrac{8}{3} X_2 \tag{53}$$

with

$$X_1 = \frac{8}{3}\epsilon^2 + \frac{8}{3}\epsilon u + u^2 \qquad \text{and} \qquad X_2 = \frac{3}{2\sqrt{2}} \tan^{-1} \frac{4\epsilon + 3u}{2\sqrt{2}\epsilon}.$$

Plots of the solutions, Eqs. (32), (45) and (53) are being prepared and are to be presented in a companion paper in the near future. In addition the solutions will be compared to those obtained numerically.

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